

A Stability Theorem for Maximal K_{r+1} -free Graphs

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Abstract

For $r \geq 2$, we show that every maximal K_{r+1} -free graph G on n vertices with $(1 - \frac{1}{r})\frac{n^2}{2} - o(n^{\frac{r+1}{r}})$ edges contains a complete r -partite subgraph on $(1 - o(1))n$ vertices. We also show that this is best possible. This result answers a question of Tyomkyn and Uzzell.

1 Introduction

For a positive integer $r \geq 2$, a graph G is said to be $(r + 1)$ -saturated (or *maximal K_{r+1} -free*) if it contains no copy of K_{r+1} , but the addition of any edge from the complement \overline{G} creates at least one copy of K_{r+1} . Let $T_r(n)$ denote the *Turán graph*, the n -vertex, complete r -partite graph for which each of the r classes is of order $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. We write $t_r(n) = e(T_r(n))$, and note that $t_r(n) = (1 - \frac{1}{r})\frac{n^2}{2} + O(n)$. Whenever we speak of an r -partite subgraph, we require that they are induced. This is to exclude any mention of the degenerate (not induced) r -partite subgraphs where $r - 1$ classes are empty and one class is very large.

The classical theorem of Turán [12] tells us that, for an integer $r \geq 2$, the maximum number of edges in a graph not containing a K_{r+1} is $t_r(n)$, and that $T_r(n)$ is the unique K_{r+1} -free graph obtaining this maximum. Erdős and Simonovits [5, 7, 10] discovered that this extremal problem exhibits a certain “stability” phenomenon: K_{r+1} -free graphs for which $e(G)$ is “close” to $t_r(n)$ must resemble the Turán graph in an appropriate sense. In particular, they proved that every n -vertex, K_{r+1} -free graph with at least $t_r(n) - o(n^2)$ edges can be transformed into $T_r(n)$ by making at most $o(n^2)$ edge deletions and additions.

Beyond the seminal work of Erdős and Simonovits, we are lead to consider finer aspects of this phenomenon. Most generally, it is natural to ask how the structure of a K_{r+1} -free graph G comes to resemble the Turán graph, as the number of edges $e(G)$ approaches the Turán number $t_r(n)$. This question is the object of study in several papers [1, 4, 8, 9, 11, 13] (even if implicitly) and is continued here, where we determine the edge threshold for which every $(r + 1)$ -saturated graph contains an “almost spanning” Turán subgraph.

One such result regarding the finer structure of stability is due to Brouwer [4] (and co-discovered by [1, 8, 9, 11]), who showed that if the number of edges of an $(r+1)$ -saturated graph G is roughly within $\frac{n}{r}$ of the Turán number $t_r(n)$, then G is a complete r -partite graph (restated below as Theorem 2.2). Continuing this line of investigation, Tyomkyn and Uzzell [13] proved that every 4-saturated graph on n vertices and with $t_3(n) - cn$ edges contains a complete 3-partite graph on $(1 - o(1))n$ vertices. They went on to ask if one can similarly find “almost-spanning”, complete r -partite subgraphs in $(r+1)$ -saturated graphs with many edges, for $r \geq 4$. The main result of this paper is to resolve the question of Tyomkyn and Uzzell, in a stronger form. Not only do we show that this phenomenon persists for $(r+1)$ -saturated graphs for all $r \geq 2$, but we also determine the edge threshold for which the result fails to hold. In particular, we show the following.

Theorem 1.1. *Let $r \geq 2$ be an integer. Every $(r+1)$ -saturated graph G on n vertices with $t_r(n) - o(n^{\frac{r+1}{r}})$ edges contains a complete r -partite subgraph on $(1 - o(1))n$ vertices.*

We also show that this theorem is tight in the sense that for every $\delta > 0$ there exist graphs G with $t_r(n) - \delta n^{\frac{r+1}{r}}$ edges for which the conclusion of Theorem 1.1 fails.

We actually deduce Theorem 1.1 from a stronger, quantitative result, which we now make precise. For a graph G and an integer $r \geq 2$, define the graph parameter

$$g_r(G) = \min\{|T| : T \subseteq V(G), G - T \text{ is complete } r\text{-partite}\}.$$

For $n, m \in \mathbb{N}$, let $\mathcal{S}_r(n, m)$ denote the set of all $(r+1)$ -saturated graphs on n vertices with at least $t_r(n) - m$ edges. Then define $g_r(n, m) = \max\{g_r(G) : G \in \mathcal{S}_r(n, m)\}$. The quantitative form of our main theorem gives a determination of the function $g_r(n, m)$, up to constants, for m such that $(r-1)n \leq m \leq n^{\frac{r+1}{r}}$, and some modest conditions on n .

Theorem 1.2. *Let r, n be integers satisfying $r \geq 2$, $n \geq 2^{6r}$, and let $(r-1)n^{-1} \leq \varepsilon \leq n^{-\frac{r-1}{r}}$. Every $(r+1)$ -saturated graph with $t_r(n) - \varepsilon n^2$ edges contains a complete r -partite subgraph on $(1 - C_r \varepsilon n^{\frac{r-1}{r}})n$ vertices, where C_r is a constant depending only on r .*

We also give constructions showing that this result is tight, up to the value of C_r . In other words, if n, r, ε are as above, then

$$c_r \varepsilon n^{2-1/r} \leq g_r(n, \varepsilon n^2) \leq C_r \varepsilon n^{2-1/r},$$

where $C_r, c_r > 0$ are constants depending only on r .

We point out that this explicit form of our main result takes a major step towards a further question of Tyomkyn and Uzzell [13], who asked for the determination of $g_3(n, cn)$.

We also consider the situation for $(r+1)$ -saturated graphs with $t_r(n) - Cn^{\frac{r+1}{r}}$ edges; that is, just beyond the edge threshold in Theorem 1.1. In this range it is perhaps most natural to consider

balanced r -partite subgraphs.¹ For a graph G and integer $r \geq 2$, define $g_r^*(G) = \min\{|T| : G - T \text{ is balanced complete } r\text{-partite}\}$ and, for $m, n \in \mathbb{N}$, define $g_r^*(n, m) = \max\{g_r^*(G) : G \in \mathcal{S}_r(n, m)\}$. Thus, $g_r^*(n, m)$ is the maximum number of vertices one is required to delete from an $(r+1)$ -saturated graph on n vertices with at least $t_r(n) - m$ edges such that the remaining graph is balanced complete r -partite. While it is easy to show (given Theorem 1.2) that $g_r^*(n, m) = o(n)$ when $m = o(n^{\frac{r+1}{r}})$, we show that $g_r^*\left(n, Cn^{\frac{r+1}{r}}\right) \geq \left(1 - \frac{c'r^{5/2}}{C^{1/2}}\right)n$, for an absolute constant c' and n sufficiently large.

Theorem 1.3. *Let $r \geq 2$ be an integer and let $\delta > 0$. There exists a constant $C = C(r, \delta)$ such that, for n sufficiently large, there exists an n -vertex $(r+1)$ -saturated graph G with no balanced complete r -partite subgraph on more than δrn vertices and $e(G) \geq t_r(n) - Cn^{\frac{r+1}{r}}$.*

We do not have any corresponding upper bounds on $g_r^*(n, m)$ in this range of m .

The rest of the paper is organized as follows. In Section 2, we prove our main result, Theorem 1.2. Roughly speaking, we first show that any K_{r+1} -free graph with many edges has a rather substantial r -partite subgraph. We then show that one can refine this resultant r -partite graph by making each bipartite graph between partition classes complete, while removing relatively few vertices. In Section 3, we provide the aforementioned constructions which exhibit the tightness of Theorems 1.1 and 1.2; in Section 4, we prove Theorem 1.3. Finally, in Section 5 we state some further questions.

2 The Proof of Theorem 1.2

Our notation is mostly standard (see, for example, [3]). For a subset $S \subseteq V(G)$ we denote by $N_G(S) = \bigcap_{v \in S} N_G(v)$ the common (or joint) neighbourhood of S in G . We shall omit the subscript ' G ' if the underlying graph is understood. If X_1, \dots, X_r are disjoint subsets of $V(G)$, we denote by $G[X_1, \dots, X_r]$ the r -partite graph induced in G with vertex classes X_1, \dots, X_r . All other notation we need shall be introduced as necessary.

2.1 Preliminary lemmas

Let us now work towards establishing Theorem 1.2. For that we state and prove two lemmas, the second of which is the core of the proof. For the first lemma we use the following theorem of Andrásfai, Erdős, and Sós [2], although the precise value of the constant $\frac{3r-4}{3r-1}$ is unimportant for us; we only need that it is strictly less than the Turán density.

Theorem 2.1. *For $r \geq 2$ let G be a K_{r+1} -free graph on n vertices which is not r -partite. Then there is a vertex v of G with*

$$d(v) \leq \frac{3r-4}{3r-1}n.$$

We shall also use the following result of Brouwer [4], mentioned in the introduction.

¹Recall that an r -partite graph with vertex classes V_1, \dots, V_r is *balanced* if $||V_i| - |V_j|| \leq 1$ for all $i, j \in [r]$.

Theorem 2.2. *Let $r \geq 2$, $n \geq 2r + 1$, and let G be an K_{r+1} -free, n -vertex graph. If $e(G) \geq t_r(n) - \lfloor \frac{n}{r} \rfloor + 2$, then G is r -partite.*

Here, then, is our first lemma, which grants us a sizable induced r -partite subgraph.

Lemma 2.3. *For $r \geq 2$ there is a constant d_r , depending only on r , such that the following holds. Let $n \geq 4r$ and $0 \leq \varepsilon \leq (10r^2(3r - 1))^{-1}$. If G is an n -vertex K_{r+1} -free graph with $e(G) \geq t_r(n) - \varepsilon n^2$, then there is a subset $T \subseteq V(G)$ with $|T| \leq d_r \varepsilon n$ such that $G - T$ is r -partite.*

Proof. If $\varepsilon < (2rn)^{-1}$, then $e(G) > t_r(n) - \frac{n}{2r} \geq t_r(n) - \lfloor \frac{n}{r} \rfloor + 1$, where the second inequality follows by our assumption that $n \geq 4r$. Therefore by Theorem 2.2, G is r -partite, and there is nothing to prove. Accordingly, we may assume $\varepsilon \geq (2rn)^{-1}$.

Set $G_1 = G$. Suppose that G_1, \dots, G_i have been defined for some $i \in [n]$. If G_i is not r -partite then pick a vertex $v_i \in V(G_i)$ with $d_{G_i}(v_i) \leq \frac{3r-4}{3r-1}|G_i|$ according to Theorem 2.1. Set $G_{i+1} = G_i - v_i$. Suppose this process terminates at stage $t \in [n]$. Then $G_{t+1} = G - \{v_1, \dots, v_t\}$ is r -partite. We claim that $t \leq d_r \varepsilon n$ for some constant d_r depending only on r . This follows from a simple calculation. Indeed, as $e(G_{t+1}) \leq \frac{r-1}{2r}(n-t)^2$ by Turán's theorem, we have

$$\begin{aligned} e(G) &\leq \frac{3r-4}{3r-1} \left(n + (n-1) + \dots + (n-t+1) \right) + \frac{r-1}{2r}(n-t)^2 \\ &= \frac{3r-4}{3r-1} \left(nt - \binom{t}{2} \right) + \frac{r-1}{2r}(n-t)^2, \end{aligned}$$

and using the lower bound on $e(G)$ we obtain

$$t_r(n) - \frac{r-1}{2r}(n-t)^2 + \frac{3r-4}{3r-1} \binom{t}{2} \leq \frac{3r-4}{3r-1} nt + \varepsilon n^2. \quad (1)$$

Further, using the trivial lower bound $t_r(n) \geq (1 - 1/r) \binom{n}{2}$ applied to (1) and rearranging yields the equivalent inequality

$$t \left(1 - \frac{t}{2n} - \frac{r(3r-4)}{2n} \right) - \frac{1}{2}(r-1)(3r-1) \leq r(3r-1)\varepsilon n,$$

which is easily seen to fail if $t = 10r^2(3r-1)\varepsilon n$ when $(2rn)^{-1} \leq \varepsilon \leq (10r^2(3r-1))^{-1}$. Therefore, Lemma 2.3 holds with some constant $d_r < 10r^2(3r-1)$. \square

The next lemma is the heart of the proof of our main theorem. Before stating it we introduce some notation and a bit of terminology. If G is an r -partite graph with vertex partition V_1, \dots, V_r , then we denote by $\tilde{G}[V_1, \dots, V_r]$ the r -partite complement of G with respect to the partition V_1, \dots, V_r . In other words $\tilde{G}[V_1, \dots, V_r]$ has vertex set $V_1 \cup \dots \cup V_r$ and its edges are precisely the non-edges of G which join two vertices belonging to distinct vertex classes of V_1, \dots, V_r . Often we simply speak of *the r -partite complement* in the case that the vertex partition we are using is clear from context, and we shall simply write \tilde{G} . We say that a subset $S \subseteq V(G)$ of the vertices of a graph G *covers* an edge e if at least one of the endpoints of e lies in S . Further, we let $I_G(S)$ denote

the collection of edges of G covered by S . An r -saturating edge in G is an edge of the complement \overline{G} the addition of which creates a copy of K_r in G . If $X, Y \subseteq V(G)$ are subsets of vertices, then we say that a non-edge e is an r -saturating (X, Y) edge if it is r -saturating with one endpoint in X , and the other in Y . A K_r -matching in a graph G is a collection of vertex disjoint copies of K_r in G . Lastly, before stating and proving the lemma, let us collect a simple observation that will be of use.

Observation 2.4. *Suppose that G is a bipartite graph with vertex classes V_1 and V_2 with $e(G) = \alpha|V_1||V_2|$, where $\alpha \in [0, 1]$. Then for any $1 \leq t \leq |V_2|$ there is a subset $W \subseteq V_2$ of size t such that the induced graph on $V_1 \cup W$ has at least $\alpha|V_1|t$ edges.*

Proof. This assertion follows from a simple averaging argument. For $Y \subseteq V_2$ let $e(V_1, Y)$ denote the number of edges of G with an endpoint in Y . Then

$$\sum_{Y \in V_2^{(t)}} e(V_1, Y) = e(G) \binom{|V_2| - 1}{t - 1} = \alpha|V_1|t \binom{|V_2|}{t},$$

so there exists a subset $W \in V_2^{(t)}$ with $e(V_1, W) \geq \alpha|V_1|t$. \square

Lemma 2.5. *Let $r \geq 2$ be an integer and let G be a K_r -free, r -partite graph with vertex classes $A, B, X_1, \dots, X_{r-2}$. Then the following statements hold.*

1. *There is a subset $R \subseteq A \cup B$ that covers all r -saturating (A, B) edges in G and*

$$|I_{\tilde{G}}(R)| \geq c_r |R|^{\frac{r}{r-1}},$$

for some constant $c_r > 0$ depending only on r .

2. *Suppose that $t \geq 1$ is an integer with $r - t \geq 2$, that E is a collection of non-edges between A, B , and that there exist K_{r-t} -free subgraphs $H_1, \dots, H_s \subseteq G$ such that every non-edge of E is $(r-t)$ -saturating in at least one of the graphs H_1, \dots, H_s . Then there exists a set $R' \subseteq A \cup B$ covering all non-edges of E with*

$$|I_{\tilde{G}}(R')| \geq c'_{r-t} s^{-\frac{1}{r-t-1}} |R'|^{\frac{r-t}{r-t-1}},$$

where c'_{r-t} is a constant depending only on r, t .

Proof. We prove these two statements simultaneously by induction on r . The case $r = 2$ is trivial: G must be empty. The first part holds by simply choosing the smaller of the two parts of the bipartite graph G and the second part of the statement is vacuous as there is no appropriate choice for t .

So, assuming that the result holds for $r - 1 \geq 2$, we prove it for r . To this end, let G be a K_r -free, r -partite graph with vertex sets $A, B, X_1, \dots, X_{r-2}$. We start with the proof of Part 2 as we shall need it to prove Part 1.

Proof of Part 2: Suppose we are given a collection E of non-edges between A, B and subgraphs H_1, \dots, H_s satisfying the requirements of the lemma. Start by enumerating the collection of subgraphs

$$\{H_i[A \cup B \cup X_{i_1} \cup \dots \cup X_{i_{r-t-2}}] : i \in [s], 1 \leq i_1 < \dots < i_{r-t-2} \leq r-2\}$$

by $H'_1, \dots, H'_{s'}$, where $s' \leq \binom{r-2}{r-t-2}s$. We now iteratively apply induction inside each of the graphs $H'_1, \dots, H'_{s'}$: at each stage we remove a set granted by the induction hypothesis before moving to the next graph in the enumeration.

We shall define a sequence of disjoint subsets $R_1, \dots, R_{s'}$ of $A \cup B$ with the following properties:

1. $G_1 = G$ and $G_{i+1} = G_i - R_i$ for all $i \geq 1$.
2. $|I_{\tilde{G}_i}(R_i)| \geq c_{r-t}|R_i|^{\frac{r-t}{r-t-1}}$ for each $i \geq 1$, where c_{r-t} is the constant given by the induction hypothesis of the lemma (here, the r -partite complement \tilde{G}_i is with respect to the ‘obvious’ r -partition of G_i).
3. Every non-edge of E is covered by $R_1 \cup \dots \cup R_{s'}$.

Suppose that, for $i \in [s' - 1]$, the graphs G_1, \dots, G_i have been defined. Apply the induction hypothesis of Lemma 2.5 to the $(r-t)$ -partite, K_{r-t} -free graph $H'_i \cap G_i$ to find a set $R_i \subseteq V(H'_i \cap G_i) \cap (A \cup B)$ with $|I_{\tilde{G}_i}(R_i)| \geq c_{r-t}|R_i|^{\frac{r-t}{r-t-1}}$ that covers all $(r-t)$ -saturating (A, B) edges in H'_i . Finally set $G_{i+1} = G_i - R_i$. To check that every non-edge of E is covered by $R_1 \cup \dots \cup R_{s'}$, simply recall that we assumed that every non-edge of E is $(r-t)$ -saturating in one of the subgraphs H_1, \dots, H_s and therefore $(r-t)$ -saturating in one of the subgraphs $H'_1, \dots, H'_{s'}$. Thus, a non-edge $e \in E$ is $(r-t)$ -saturating in some H'_j for some $j \in [s']$, and so it will be covered by one of R_1, \dots, R_j . That is, it will be covered in stage j , if it has not been covered already.

To finish the proof of Part 2 of the lemma, we write $R' = R_1 \cup \dots \cup R_{s'}$. Noting that the sets $R_1, \dots, R_{s'}$ are pairwise disjoint, we apply Hölder’s inequality to obtain

$$|R'| = \sum_{i=1}^{s'} |R_i| \leq s'^{\frac{1}{r-t}} \left(\sum_{i=1}^{s'} |R_i|^{\frac{r-t}{r-t-1}} \right)^{\frac{r-t-1}{r-t}},$$

and therefore

$$s'^{-\frac{1}{r-t-1}} |R'|^{\frac{r-t}{r-t-1}} \leq \sum_{i=1}^{s'} |R_i|^{\frac{r-t}{r-t-1}}.$$

Now, since the sets of edges $\{I_{\tilde{G}_i}(R_i)\}_{i \in [s']}$ are pairwise disjoint, we may estimate

$$\begin{aligned} |I_{\tilde{G}}(R')| &\geq \sum_{i=1}^{s'} |I_{\tilde{G}_i}(R_i)| \geq \sum_{i=1}^{s'} c_{r-t} |R_i|^{\frac{r-t}{r-t-1}} \\ &\geq c_{r-t} s'^{-\frac{1}{r-t-1}} |R'|^{\frac{r-t}{r-t-1}} \\ &\geq c'_{r-t} s^{-\frac{1}{r-t-1}} |R'|^{\frac{r-t}{r-t-1}}, \end{aligned}$$

where c'_{r-t} is a constant depending only on r, t . This completes the proof of Part 2 of Lemma 2.5. To prove the first part we use the second part along with an extra ingredient.

Proof of Part 1 : We may assume that there is *some* saturating (A, B) -edge, otherwise we are trivially done with the choice of $R = \emptyset$. So, let \mathcal{M} be a K_{r-2} -matching of maximum size in the graph $G[X_1, \dots, X_{r-2}]$ and let Y denote the collection of vertices contained in a clique of \mathcal{M} . Note that \mathcal{M} is nonempty as there is some saturating (A, B) -edge, and put $L = |Y| = (r-2)|\mathcal{M}| > 0$. For each $y \in Y$, let $G(y)$ be the $(r-1)$ -partite graph induced on the neighbourhood of y in G with vertex classes $N(y) \cap A, N(y) \cap B$ along with $N(y) \cap X_i$ for $y \notin X_i, i \in [r-2]$.

Now, observe that, by the maximality of \mathcal{M} , every r -saturating (A, B) edge is $(r-1)$ -saturating in one of the graphs $\{G(y)\}_{y \in Y}$. Hence we may apply the bound in Part 2 of the lemma to obtain a set R_0 which covers every r -saturating (A, B) edge and

$$|I_{\tilde{G}}(R_0)| \geq c'_{r-1} L^{-\frac{1}{r-2}} |R_0|^{\frac{r-1}{r-2}}. \quad (2)$$

However, this bound is not useful if L is too large. But if L is large we expect a random subset of $A \cup B$ to cover quite a few non-edges between $A \cup B$ and Y . So we randomly augment R_0 with an additional set of $|R_0|$ vertices to form R . This set R will only be at most a constant factor larger than R_0 , but will cover enough non-edges between $A \cup B$ and Y to make up for the loss in the above estimate.

We first show that we can assume that there are not too many edges between Y and $A \cup B$. For each $K \in \mathcal{M}$ and $S \subseteq V(G)$ we denote by $d_S(K)$ the number of vertices of S joined to every vertex of K . We may assume that, for every $K \in \mathcal{M}$, either $d_A(K) \leq \frac{1}{2}|A|$ or $d_B(K) \leq \frac{1}{2}|B|$. Indeed, suppose that there is $K \in \mathcal{M}$ with $d_A(K) > \frac{1}{2}|A|$ and $d_B(K) > \frac{1}{2}|B|$. As G is K_r -free we must then count more than $\frac{1}{4}|A||B|$ non-edges between A and B . Setting R to be the smaller of A and B , we see that trivially R covers all r -saturating (A, B) edges and

$$|I_{\tilde{G}}(R)| > \frac{1}{4}|A||B| \geq \frac{1}{4}|R|^2,$$

so we are done (with room to spare). Therefore, we may assume that for every $K \in \mathcal{M}$ either $d_A(K) \leq \frac{1}{2}|A|$ or $d_B(K) \leq \frac{1}{2}|B|$.

Write $\mathcal{M} = \mathcal{M}_A \cup \mathcal{M}_B$, where \mathcal{M}_A are those $K \in \mathcal{M}$ which satisfy $d_A(K) \leq \frac{1}{2}|A|$ and \mathcal{M}_B are those that satisfy $d_B(K) \leq \frac{1}{2}|B|$. Then, without loss of generality, we have $|\mathcal{M}_A| \geq \frac{1}{2}|\mathcal{M}|$. Now since each $K \in \mathcal{M}_A$ sends at least $\frac{1}{2}|A|$ non-edges to A and since each clique in \mathcal{M} is vertex-disjoint,

we have that there are at least $\frac{1}{4}|A||\mathcal{M}| = \frac{1}{4(r-2)}|A||Y|$ non-edges between Y and A .

Now, we may assume that $|R_0| \leq |A|$. Indeed, suppose otherwise that $|R_0| > |A|$. If $|A||\mathcal{M}| \geq |A|^{\frac{r}{r-1}}$, we are done by choosing $R = A$, since then $|I_{\tilde{G}}(A)| \geq \frac{1}{4}|A||\mathcal{M}| \geq \frac{1}{4}|A|^{\frac{r}{r-1}}$. Otherwise, $|\mathcal{M}| < |A|^{\frac{1}{r-1}} < |R_0|^{\frac{1}{r-1}}$, and using (2) yields $|I_{\tilde{G}}(R_0)| \geq c'_r |R_0|^{\frac{r}{r-1}}$, so we are done with the choice $R = R_0$.

Hence, assuming that $|R_0| \leq |A|$, by Observation 2.4, one can find a subset $R'_0 \subseteq A$ of size $|R_0|$ such that the number of non-edges between R'_0 and Y is at least $\frac{1}{4(r-2)}|R_0||Y| = \frac{1}{4}|R_0||\mathcal{M}|$.

We now set $R = R_0 \cup R'_0$ and claim that R is our desired set. First note that R covers all r -saturating (A, B) edges in G , as R_0 already does. To count the number of non-edges covered by R , we note that $|R| \leq 2|R_0|$, and so we have

$$\begin{aligned} 2|I_{\tilde{G}}(R)| &\geq |I_{\tilde{G}}(R_0)| + |I_{\tilde{G}}(R'_0)| \\ &\geq c'_{r-1} L^{-\frac{1}{r-2}} |R_0|^{\frac{r-1}{r-2}} + \frac{1}{4}|R_0||\mathcal{M}| \\ &\geq c' |\mathcal{M}|^{-\frac{1}{r-2}} |R|^{\frac{r-1}{r-2}} + \frac{1}{8}|R||\mathcal{M}|, \end{aligned} \tag{3}$$

where $c' = c'_{r-1} 2^{-\frac{r-1}{r-2}} (r-2)^{-\frac{1}{r-2}}$. A trivial analysis reveals that the quantity on the right-hand side of (3) is minimized in $|\mathcal{M}|$ if $|\mathcal{M}| = (8c'/(r-2))^{\frac{r-2}{r-1}} |R|^{\frac{1}{r-1}}$. Substituting this value of $|\mathcal{M}|$ back into (3) yields

$$|I_{\tilde{G}}(R)| \geq c_r |R|^{\frac{r}{r-1}},$$

where c_r is a constant depending only on r . □

2.2 Finishing the proof

We can now proceed to the proof of Theorem 1.2. Let us state the precise quantitative result that we prove, which is slightly stronger than what is stated in the Introduction.

Theorem 2.6. *Let r, n be integers satisfying $r \geq 2$ and $n \geq (10r^2(3r-1))^2$. Then every $(r+1)$ -saturated graph G on n vertices with at least $t_r(n) - \varepsilon n^2$ edges contains a complete r -partite subgraph on at least $\left(1 - C_r \varepsilon n^{\frac{r-1}{r}}\right) n$ vertices, where C_r is a constant depending only on r .*

Proof. Suppose that G is an n -vertex $(r+1)$ -saturated graph with $e(G) \geq t_r(n) - \varepsilon n^2$, as in the statement. Note that we may insist that the constant C_r is at least 1 in Theorem 2.6: increasing the value of C_r only makes the theorem easier. The result is then trivial if $\varepsilon > n^{-\frac{r-1}{r}}$ and so we may assume that $\varepsilon \leq n^{-\frac{r-1}{r}}$. Since $n \geq (10r^2(3r-1))^2$ we have that $\varepsilon \leq (10r^2(3r-1))^{-1}$, so we may apply Lemma 2.3 to obtain a subset $T \subseteq V(G)$ such that $|T| \leq d_r \varepsilon n$ and $G - T$ is r -partite. Let the vertex classes of $G - T$ be V_1, \dots, V_r . We now simply apply Part 2 of Lemma 2.5 to common neighbourhoods of appropriate subsets of T . But before we do this we need a bound on $e(\tilde{G}[V_1, \dots, V_r])$, the number of non-edges between the parts V_1, \dots, V_r .

We claim that $e(\tilde{G}[V_1, \dots, V_r]) \leq (d_r + 1)\varepsilon n^2$. To see this, first note that if $|T| = 0$, then G is r -partite and $e(\tilde{G}[V_1, \dots, V_r]) \leq \varepsilon n^2$. So, we may assume that $|T| \geq 1$. In this case, the number of

non-edges in G is at most $\binom{n}{2} - t_r(n) + \varepsilon n^2$, and at least

$$\sum_{i=1}^r \binom{|V_i|}{2} + e(\tilde{G}[V_1, \dots, V_r]) \geq r \binom{\frac{n-|T|}{r}}{2} + e(\tilde{G}[V_1, \dots, V_r]),$$

by convexity of the function $x \mapsto \binom{x}{2}$. By using the estimate $t_r(n) \geq (1 - \frac{1}{r}) \binom{n}{2}$ and rearranging, we get

$$e(\tilde{G}[V_1, \dots, V_r]) \leq \varepsilon n^2 + \frac{1}{r} \binom{n}{2} - r \binom{\frac{n-|T|}{r}}{2} < \varepsilon n^2 + \frac{r-1}{2r} n + \frac{n|T|}{r} \quad (4)$$

$$= \varepsilon n^2 + 2n|T| \left(\frac{r-1}{4r|T|} + \frac{1}{2r} \right). \quad (5)$$

Now, if $|T| \geq r/2$, then (5) is at most $\varepsilon n^2 + \frac{2n|T|}{r}$, and we are done. If $|T| < r/2$, then by (4) we have $e(\tilde{G}[V_1, \dots, V_r]) < \varepsilon n^2 + n = (1 + \frac{1}{\varepsilon n}) \varepsilon n^2$. But clearly $\frac{1}{\varepsilon n} \leq d_r$, as otherwise $|T| < 1$. Hence, the desired bound on $e(\tilde{G}[V_1, \dots, V_r])$ holds.

For $t \in [r-1]$ let \mathcal{C}_t denote the collection of copies of K_t contained in $G[T]$, the graph induced on T . We say a non-edge e is of *type* t if it lies between two of the classes V_1, \dots, V_r , and the addition of e to G creates a K_{r+1} with exactly t vertices in T . Since G is $(r+1)$ -saturated and $G[V_1, \dots, V_r]$ is a K_{r+1} -free graph, every non-edge between two of the classes V_1, \dots, V_r is of type t for some $t \in [r-1]$. For $t \in [r-1]$ we let E_t denote the collection of type t non-edges.

Set $V = V_1 \cup \dots \cup V_r$ and define $\mathcal{G}_t = \{G[N(K) \cap V] : K \in \mathcal{C}_t\}$ for $t \in [r-1]$. For each $i \neq j \in [r]$, we show that one can make the induced bipartite graph $G[V_i, V_j]$ complete by removing a relatively small number of vertices. Doing this in succession for each of the $\binom{r}{2}$ pairs V_i, V_j with $i \neq j$ then yields a complete r -partite subgraph.

So fix $i \neq j \in [r]$ and note that for each $t \in [r-1]$, each graph in the collection \mathcal{G}_t is K_{r+1-t} -free and every (V_i, V_j) non-edge of E_t is $(r+1-t)$ -saturating in one of the graphs of \mathcal{G}_t . So for each $t \in [r-1]$ we may invoke Lemma 2.5 to obtain a set $S_t(i, j) \subseteq V_i \cup V_j$ that covers every $(r+1)$ -saturating (V_i, V_j) edge of type t and

$$|I_{\tilde{G}[V_1, \dots, V_r]}(S_t(i, j))| \geq c'_{r+1-t} |\mathcal{C}_t|^{-\frac{1}{r-t}} |S_t(i, j)|^{\frac{r+1-t}{r-t}}.$$

Moreover, $|I_{\tilde{G}[V_1, \dots, V_r]}(S_t(i, j))| \leq e(\tilde{G}[V_1, \dots, V_r]) \leq (d_r + 1) \varepsilon n^2$, and using the trivial bound $|\mathcal{C}_t| \leq (d_r \varepsilon n)^t$, we obtain

$$|S_t(i, j)| \leq C_{r,t} \left(\varepsilon^{\frac{r}{r+1-t}} n^{\frac{r-1}{r+1-t}} \right) n,$$

where $C_{r,t}$ is a constant depending only on r, t , for each $t \in [r-1]$, and $i \neq j \in [r]$.

As every edge between the parts V_1, \dots, V_r is of type t for some $t \in [r-1]$, we conclude that the set $S = \bigcup_{t=1}^{r-1} \bigcup_{i \neq j \in [r]} S_t(i, j)$ covers every non-edge between the parts V_1, \dots, V_r . It follows that

$G - S - T$ is a complete r -partite graph. To bound $|S|$ recall that $\varepsilon \leq n^{-\frac{r-1}{r}}$. Then we have

$$\begin{aligned} |S| &\leq \sum_{t=1}^{r-1} \sum_{i \neq j \in [r]} C_{r,t} \left(\varepsilon^{\frac{r}{r+1-t}} n^{\frac{r-1}{r+1-t}} \right) n \\ &\leq (r-1) \binom{r}{2} \max_{t \in [r-1]} \{C_{r,t}\} \left(\varepsilon n^{\frac{r-1}{r}} \right) n \\ &\leq C'_r \left(\varepsilon n^{\frac{r-1}{r}} \right) n, \end{aligned}$$

where the constant C'_r depends only on r . It is here that we have used the condition $\varepsilon \leq n^{-\frac{r-1}{r}}$, since this implies that the dominating term in the sum above is the one with $t = 1$. Hence we have found a complete r -partite subgraph on

$$n - |S| - |T| \geq n - C'_r \left(\varepsilon n^{\frac{r-1}{r}} \right) n - d_r \varepsilon n \geq \left(1 - C_r \varepsilon n^{\frac{r-1}{r}} \right) n$$

vertices, for some constant C_r . This completes the proof. \square

3 Constructions

The aim of this section is to describe two families of constructions that demonstrate the optimality of Theorem 1.1. One reason for giving two different constructions is to impress upon the reader that there are indeed different constructions that exhibit the tightness of the “soft” version (Theorem 1.1) of our main result. Also, the first construction is easier to digest while the second construction will show that the more precise quantitative bound in Theorem 1.2 is tight up to the constant C_r .

3.1 First construction

For our first construction, we define a graph G , for each $r \geq 2$ and sufficiently large n , that is an $(r+1)$ -saturated graph on n vertices, with at least $t_r(n) - cn^{\frac{r+1}{r}}$ edges and with no complete r -partite subgraph on more than $(1 - \frac{1}{r})n$ vertices.

Let $r, m, t \in \mathbb{N}$ be such that $r \geq 2, t \geq 3, m \geq t^{r-1}$, and that $t^{r-1} | m$. We define G to be a $2r$ -partite graph on $n = rm + (r-1)t + 1$ vertices with vertex partition

$$V(G) = (V_1 \cup \dots \cup V_{r-2}) \cup (Y_1 \cup Y_2) \cup (X_1 \cup \dots \cup X_{r-1}) \cup \{x^*\},$$

where $|V_1| = |V_2| = \dots = |V_{r-2}| = |Y_1| = |Y_2| = m$ and $|X_1| = \dots = |X_{r-1}| = t$ and x^* is a single vertex. We additionally equip each of the sets Y_1, Y_2 , with a partition into t^{r-1} “cells” of equal size. We index each cell $Y_i(v)$ of Y_i , by a vector $v \in [t]^{r-1}$ and write

$$Y_i = \bigcup_{v \in [t]^{r-1}} Y_i(v),$$

for $i \in \{1, 2\}$. We also regard the sets X_1, \dots, X_{r-1} as endowed with linear orderings, so we may speak of “the i th element of X_j ”, provided $i \in [t], j \in [r-1]$.

Turning our attention to the edges of G , we declare each of the following pairs to induce complete bipartite subgraphs: $\{(V_i, V_j)\}_{i \neq j}, \{(X_i, X_j)\}_{i \neq j}, \{(Y_i, V_j)\}_{i=1,2; j \in [r-2]}$ and $\{(X_i, V_j)\}_{i \neq j; i, j \in [r-2]}$. Now, we add an edge between $y_1 \in Y_1, y_2 \in Y_2$ if and only if the cells that they are contained in *do not* receive the same index $v \in [t]^{r-1}$. We define the adjacencies between vertices in X_1, \dots, X_{r-1} and Y_1, Y_2 thus: for each $k \in [t], l \in [r-1]$, we join the k th vertex in X_l to all vertices in the cells $Y_1((i_1, i_2, \dots, i_{r-1})), Y_2((i_1, i_2, \dots, i_{r-1}))$, where $i_l = k$ and $i_1, \dots, i_{r-1} \in [t]$. Finally, we join the vertex x^* to all vertices in $X_1 \cup \dots \cup X_{r-1} \cup V_1 \cup \dots \cup V_{r-2}$.

It is not hard, though slightly tedious, to check that this graph is $(r+1)$ -saturated. We now set $t = m^{\frac{1}{r}}$ and obtain that $e(G) > t_r(n) - (r^{\frac{r-1}{r}} + o(1))n^{\frac{r+1}{r}}$. Indeed, first count the number of missing edges between the classes Y_1, Y_2 , where each vertex in Y_1 is not joined to each vertex in Y_2 in the corresponding cell. This means that we count $m^{\frac{m}{t^{r-1}}} = m^{\frac{r+1}{r}} \leq r^{-\frac{r+1}{r}} n^{\frac{r+1}{r}}$ non-edges between the classes Y_1, Y_2 . Since $e(G) > e(G[V_1 \cup \dots \cup V_{r-2} \cup Y_1 \cup Y_2]) \geq t_r(rm) - r^{-\frac{r+1}{r}} n^{\frac{r+1}{r}} = t_r(n - ((r-1)m^{1/r} + 1)) - r^{-\frac{r+1}{r}} n^{\frac{r+1}{r}}$, and by the trivial inequalities $t_r(n-s) \geq t_r(n) - sn$ and $m \leq n/r$, one obtains

$$\begin{aligned} e(G) &> t_r(n) - \left((r-1)m^{1/r} + 1\right)n - r^{-\frac{r+1}{r}} n^{\frac{r+1}{r}} \\ &\geq t_r(n) - \left((r-1)r^{-\frac{1}{r}} + r^{-\frac{r+1}{r}} + o(1)\right)n^{\frac{r+1}{r}} \\ &\geq t_r(n) - \left(r^{\frac{r-1}{r}} + o(1)\right)n^{\frac{r+1}{r}}. \end{aligned}$$

We claim that any r -partite complete subgraph of G has order at most $(r-1)m$. As $t \ll m$, we may restrict our attention to looking for complete r -partite subgraphs in the induced graph $G[V_1 \cup \dots \cup V_{r-1} \cup Y_1 \cup Y_2]$. Obviously, a complete r -partite graph cannot include vertices from each of the cells $Y_1(v)$ and $Y_2(v)$, as there are no edges between these cells. Hence, any complete r -partite subgraph has at most $(r-1)m < (1 - \frac{1}{r})n$ vertices.

3.2 Second construction

For our main construction, we begin by inductively constructing a family of auxiliary graphs $G_{r,s}$, for each $r, s \in \mathbb{N}, r, s \geq 2$. It is useful to keep in mind that the edges of the r -partite graph $G_{r,s}$ record edges to be removed from a later graph.

Construction of $G_{r,s}$: Fix $s \in \mathbb{N}, s \geq 2$. First, we define $G_{2,s}$ to be the complete bipartite graph $K_{2s,2s}$. For $r \geq 2$, we define $G_{r+1,s}$ to be an $(r+1)$ -partite graph defined recursively as follows. Let H_1, \dots, H_s be a collection of disjoint copies of $G_{r,s}$ and suppose H_i has vertex classes A_1^i, \dots, A_r^i , for each $i \in [s]$. Define $G_{r+1,s}$ to be an $(r+1)$ -partite graph with the first r vertex classes defined as $A_j := A_j^1 \cup \dots \cup A_j^s$, for $j = 1, \dots, r$, and with the $(r+1)$ st vertex class defined as a collection of new vertices $A_{r+1} = \{x_1, \dots, x_s\}$. We define the edge set $E(G_{r+1,s}) = \bigcup_{i=1}^s E(H_i) \cup \{x_i y : y \in H_j, i, j \in [s], i \neq j\}$.

The following proposition records several useful properties of our family of graphs $G_{r,s}$.

Proposition 3.1. *The graph $G_{r,s}$ has the following properties.*

1. $G_{r,s}$ is r -partite with vertex partition $A_1 \cup \dots \cup A_r$ (and hence it makes sense to consider the r -partite complement of $G_{r,s}$ with respect to this partition).
2. $\tilde{G}_{r,s}$ is K_r -free.
3. Every edge between two different vertex classes of $G_{r,s}$ is r -saturating in $\tilde{G}_{r,s}$.
4. $|G_{r,s}| = \sum_{i=1}^{r-2} s^i + 4s^{r-1} = \frac{s}{s-1}(4s^{r-1} - 3s^{r-2} - 1) \leq 4\frac{s^r}{s-1}$.
5. $e(G_{r,s}) \leq 4(r-1)s^r$.
6. The size of the largest two vertex classes is $2s^{r-1}$.
7. There is a matching between the largest two vertex classes of $G_{r,s}$.
8. Any independent set in $G_{r,s}$ has at most $|G_{r,s}| - 2s^{r-1}$ vertices.

Proof. We shall use induction on r . The base case $r = 2$ is trivial. Suppose the assertions hold for $r \geq 2$. Clearly $G_{r+1,s}$ is $(r+1)$ -partite and $\tilde{G}_{r+1,s}$ is K_{r+1} -free. To show Part 3, notice that the only edges between vertex classes in $G_{r+1,s}$ are either inside H_i or between x_i and H_j , for some $i, j \in [s]$, $i \neq j$. If we add an edge to $\tilde{G}_{r+1,s}$ of the former type, the assertion holds simply by induction. If we add an edge $x_i y$ of the latter type, first observe that \tilde{H}_i contains a K_{r-1} , say K , that does not intersect the vertex class (among A_1, \dots, A_r) containing y . Thus, both x_i and y are joined to every vertex in K , thus forming a K_{r+1} in $\tilde{G}_{r+1,s}$. The number of vertices satisfies the relation $|G_{r+1,s}| = s + s|G_{r,s}|$ while $|G_{2,s}| = 4s$, and thus the claim follows. The number of edges satisfies the recurrence $e(G_{r+1,s}) = s \cdot e(G_{r,s}) + s(s-1)|G_{r,s}| \leq s \cdot e(G_{r,s}) + s(s-1)4\frac{s^r}{s-1} = s \cdot e(G_{r,s}) + 4s^{r+1}$ so, by induction, $e(G_{r+1,s}) \leq 4(r-1)s^{r+1} + 4s^{r+1} = 4rs^{r+1}$. Parts 6,7 follow immediately by induction. Finally, to argue Part 8, simply notice that for each $i \in [s]$ there is no independent set in H_i with more than $|H_i| - 2s^{r-1}$ vertices. Therefore, from disjointness of the H_i 's, any independent set in $G_{r+1,s}$ has at most $|G_{r+1,s}| - 2s^r$ vertices. \square

We can now proceed to construct a family of graphs that will demonstrate the tightness of Theorem 1.2.

Proposition 3.2. *Suppose that $n, r, s, t \in \mathbb{N}$ with $r, s \geq 2$ satisfy $n \geq 4s^{r-1}tr + t$. Then there exists an $(r+1)$ -saturated graph G on n vertices with $e(G) \geq t_r(n) - \frac{r-1}{r}tn - 4(r-1)ts^r$ such that any complete r -partite subgraph has at most $n - 2ts^{r-1}$ vertices.*

Proof. We let H_1, \dots, H_t be vertex disjoint copies of $G_{r,s}$ with vertex partitions $H_i = A_1^i \cup \dots \cup A_r^i$ for each $i \in [t]$. We now augment the vertex set of the H_i 's to be the vertex set for our G . First note that since $n \geq 4s^{r-1}tr + t$, we can find $l_1, \dots, l_r \in \mathbb{N}$, so that for each $j \in [r]$ the sums

$\sum_i^t |A_j^i| + l_j \in \{\lfloor \frac{n-t}{r} \rfloor, \lceil \frac{n-t}{r} \rceil\}$ and $\sum_j^r \left(\sum_i^t |A_j^i| + l_j \right) = n - t$. Note that as n is large enough, we may assume that $l_1, \dots, l_r > 0$. We now define the sets A_1, \dots, A_r as

$$A_i = A_i^1 \cup \dots \cup A_i^t \cup Y_i,$$

for $i \in [r]$, where Y_i is a collection of l_i new vertices. We additionally define $A_{r+1} = \{x_1, \dots, x_t\}$ as a collection of t new vertices and finally set $V(G) = \bigcup_{i=1}^{r+1} A_i$.

We define the edge set as follows: the vertex x_i is joined to $V(H_i)$, for each $i \in [t]$, and for $i, j \in [r]$, $x \in A_i, y \in A_j$, xy is an edge if and only if $i \neq j$ and the edge xy is *not* in any of the graphs H_1, \dots, H_t . We then add a maximal set of edges among A_{r+1} that leaves the graph K_{r+1} -free. That is, we first define a graph G' by $V(G') = V(G)$ and

$$E(G') = \{x_i y : y \in V(H_i), i \in [t]\} \cup \{xy : x \in A_i, y \in A_j, 1 \leq i < j \leq r\} \setminus \bigcup_{i=1}^t E(H_i),$$

and then augment the edge set to form $E(G)$:

$$E(G) = E(G') \cup X,$$

where $X \subseteq A_{r+1}^{(2)}$ is maximal in the sense that adding any further edge of $A_{r+1}^{(2)}$ will yield a K_{r+1} in G . We see that the final graph has

$$e(G) \geq t_r(n) - \frac{r-1}{r}tn - te(G_{r,s}) \geq t_r(n) - \frac{r-1}{r}tn - 4(r-1)ts^r.$$

We first note that any complete r -partite subgraph is of order at most $n - 2ts^{r-1}$, as for each $i \in [t]$, at most $|H_i| - 2s^{r-1}$ vertices from $V(H_i)$ can be included in a complete r -partite subgraph of G , by Part 8 of Proposition 3.1.

To see that the graph is saturated we may argue as we did in the proof of Proposition 3.1. There are only three types of edges that one could add to G : edges that lie totally within $V(H_i)$, for some $i \in [t]$, edges between A_{r+1} and one of the A_i , $i \in [r]$, and edges within a vertex class. Note that the first option must create a K_r by Proposition 3.1, which then extends to a K_{r+1} when we include x_i . If we add an edge $x_i y$, for some $y \in A_1 \cup \dots \cup A_r$, $i \in [t]$, we may form a K_{r+1} by choosing a K_{r-1} , say K , in the graph induced on $V(H_i)$ whose vertex set avoids vertices from the vertex class containing y . We then form a K_{r+1} by observing that x_i, y are joined to all of K . If we add an edge within one of the classes A_1, \dots, A_r , then we find a K_{r-1} among Y_1, \dots, Y_r that does not intersect the class that contains the added edge. Clearly this K_{r-1} is in the common neighbourhood of both points of the added edge and hence we extend to a K_{r+1} . Adding an edge within A_{r+1} guarantees a K_{r+1} by the construction of G . \square

By choosing s and t appropriately, we arrive at the following.

Theorem 3.3. For $r \geq 2$ put $b_r = \frac{r-1}{4r}$, and let $n \in \mathbb{N}$ and $\varepsilon > 0$ be such that $n \geq 2^r/b_r$ and $(r-1)n^{-1} \leq \varepsilon < \frac{r-1}{10r} (b_r n)^{-\frac{r-1}{r}}$.

1. Then there exists an $(r+1)$ -saturated graph G on n vertices with $e(G) \geq t_r(n) - \varepsilon n^2$ with no complete r -partite subgraph on more than $(1 - c_r \varepsilon n^{\frac{r-1}{r}})n$ vertices, where c_r is a constant depending only on r .
2. If we have $\varepsilon n \rightarrow \infty$, then there exists an $(r+1)$ -saturated graph on n vertices with $e(G) \geq t_r(n) - \varepsilon n^2$ and no complete r -partite subgraph on more than $(1 - (\gamma_r + o(1))\varepsilon n^{\frac{r-1}{r}})n$ vertices, where $\gamma_r = \frac{1}{2} \left(\frac{4}{(r-1)r^{r-1}} \right)^{1/r}$.

Proof. We set $s = \lfloor (b_r n)^{\frac{1}{r}} \rfloor \geq 2$ and $t = \lfloor \frac{1}{r-1} \varepsilon n \rfloor \geq 1$ in Proposition 3.2. It is easy to check that the condition $n \geq 4s^{r-1}tr + t$ holds for this choice of s and t . The resulting graph has $e(G) \geq t_r(n) - \varepsilon n^2$ and

$$\begin{aligned} g_r(G) &\geq 2ts^{r-1} \geq 2(\varepsilon n/(r-1) - 1)((b_r n)^{1/r} - 1)^{r-1} \\ &\geq \varepsilon \gamma_r n^{2-\frac{1}{r}} - 2(b_r n)^{\frac{r-1}{r}} - \varepsilon(r-1)\gamma_r b_r^{-\frac{1}{r}} n^{2-\frac{2}{r}} \\ &= \gamma_r \varepsilon n^{2-\frac{1}{r}} \left(1 - (r-1)(\varepsilon n)^{-1} - (r-1)b_r^{-\frac{1}{r}} n^{-\frac{1}{r}} \right) \end{aligned}$$

where $\gamma_r = \frac{2}{r-1} b_r^{\frac{r-1}{r}} = \frac{1}{2} \left(\frac{4}{(r-1)r^{r-1}} \right)^{1/r}$. Since $\varepsilon \geq (r-1)n^{-1}$, we have that there is no complete r -partite subgraph on more than $(1 - c_r \varepsilon n^{\frac{r-1}{r}})n$ vertices, for some constant c_r . If $\varepsilon n \rightarrow \infty$, we see that there is no complete r -partite subgraph on more than $(1 - (\gamma_r + o(1))\varepsilon n^{\frac{r-1}{r}})n$ vertices, as desired. □

4 Beyond the threshold: $(r+1)$ -saturated graphs on $t_r(n) - O(n^{\frac{r+1}{r}})$ edges

If G is an $(r+1)$ -saturated graph with $t_r(n) - o(n^{\frac{r+1}{r}})$ edges, then Theorem 1.1 tells us that G has an complete r -partite subgraph $G' = V_1 \cup \dots \cup V_r$ on $(1 - o(1))n$ vertices. It is easy to see that no two classes V_i, V_j can differ by more than $o(n)$ vertices (otherwise, there would be too few edges in G), and so we may remove at most $o(n)$ vertices to make G' balanced. In other words, there is little quantitative difference between the maximum sized *balanced*, complete r -partite subgraph and the maximum sized complete r -partite subgraph, in the edge regime $t_r(n) - o(n^{\frac{r+1}{r}})$. However, if $e(G) = t_r(n) - O(n^{\frac{r+1}{r}})$ the difference between these two problems becomes relevant, and we find it most natural to restrict our attention to *balanced* complete r -partite subgraphs.

Recall that for $n, m \in \mathbb{N}$, $g_r^*(n, m)$ is the maximum number vertices that one must remove from a $(r+1)$ -saturated graph on $t_r(n) - m$ edges so that the remaining graph is a Turán graph. In this

section, we show that $g_r^* \left(n, Cn^{\frac{r+1}{r}} \right) \geq \left(1 - \frac{c'r^{5/2}}{C^{1/2}} \right) n$, for an absolute constant c' . We remind the reader of the statement of Theorem 1.3 for convenience.

Theorem 1.3. *Let $r \geq 2$ be an integer and let $\delta > 0$. There exists a constant $C = C(r, \delta)$ such that, for n sufficiently large, there exists an n -vertex $(r+1)$ -saturated graph G with no balanced complete r -partite subgraph on more than δrn vertices and $e(G) \geq t_r(n) - Cn^{\frac{r+1}{r}}$.*

Proof. Fix $\delta > 0$ and choose $C(r, \delta) = 128r^3/\delta^2$. We let $s, t \ll n$ be integer parameters to be selected shortly. Let $T_r(n-t)$ be the Turán graph on $n-t$ vertices with vertex classes V_1, \dots, V_r . Let $V_{r+1} = \{x_1, \dots, x_t\}$ be a collection of vertices disjoint from $V(T_r(n-t))$. We shall define our graph G on vertex set $V_1 \cup \dots \cup V_{r+1}$. We fix some vertices $v_i \in V_i$ for each $i \in [r]$ and then define $V'_i = V_i \setminus \{v_i\}$. We now define an auxiliary graph H on V'_1, \dots, V'_r which will record edges that we shall delete from $T_r(n-t)$ to form our final construction.

For $i \in [t]$, let $H^i = A_1^i \cup \dots \cup A_r^i$ be t disjoint copies of the graph $G_{r,s}$ as defined in subsection 3.2 (whose vertex sets are disjoint from $V_1 \cup \dots \cup V_{r+1}$), with A_1^i, A_2^i being the two largest vertex classes (each of order $2s^{r-1}$) of H^i for each i . We shall randomly embed each H^i into $V'_1 \cup \dots \cup V'_r$ in a manner that respects the partition V'_1, \dots, V'_r .

Firstly, arbitrarily partition the family $\{H^i\}_{i \in [t]}$ into $\binom{r}{2}$ subfamilies $\mathcal{F}(p)$, indexed by pairs $p \in [r]^{(2)}$, so that each subfamily $\mathcal{F}(p)$ contains at least $\frac{t}{\binom{r}{2}} - 1$ of the graphs H^i . We now construct a collection of injections $f_i : V(H^i) \rightarrow \bigcup_{j=1}^r V'_j$ in the following way. If $H^i \in \mathcal{F}(p)$, choose any permutation σ of the elements of $[r]$ such that $\{\sigma(1), \sigma(2)\} = p$. We then define f_i randomly in stages: let $f_i|_{A_{\sigma(j)}^i} : A_{\sigma(j)}^i \rightarrow V'_j$ be an injection chosen uniformly at random for each $j \in [r]$. We define the graph $H(f_1, \dots, f_t)$ to have vertex set $V'_1 \cup \dots \cup V'_r$ and edge set $E(H(f_1, \dots, f_t)) = \bigcup_{i \in [t]} \{xy : f_i^{-1}(x)f_i^{-1}(y) \in E(H^i)\}$.

We show that the probability of making a “good” choice for these embeddings is non-zero. With foresight, we select $s = n^{\frac{1}{r}}$, $t = \frac{32r^2}{\delta^2} n^{\frac{1}{r}}$ and note that for large enough n we have $\frac{\delta}{4}n > 2s^{r-1}$. Let $E(A, B)$ be the “bad” event that the pair $A, B \subseteq V_1 \cup \dots \cup V_r$ have no edge of $H(f_1, \dots, f_t)$ between them. Letting $\mathbf{1}(E(A, B))$ denote the indicator of the event $E(A, B)$, we define the random variable

$$X = \sum_{1 \leq i < j \leq r} \sum_{A \in V_i^{(\delta n/2)}} \sum_{B \in V_j^{(\delta n/2)}} \mathbf{1}(E(A, B)),$$

which counts the number of pairs of subsets A, B of size $\delta n/2$ that has no edge between them in $H(f_1, \dots, f_t)$. To estimate the expectation of X we fix two sets (without loss of generality) $A \subseteq V_1$, $B \subseteq V_2$ each of size $\delta n/2$, and let $E_i(A, B)$, for $i \in [t]$, denote the event that $f_i(H^i)$ has no edge between A, B . Obviously, $\mathbb{P}(E(A, B)) = \prod_i \mathbb{P}(E_i(A, B))$.

We fix $i \in [t]$ and look to bound $\mathbb{P}(E_i(A, B))$. For $H^i \notin \mathcal{F}(\{1, 2\})$, we use the trivial bound $\mathbb{P}(E_i(A, B)) \leq 1$. For $H^i \in \mathcal{F}(\{1, 2\})$ we shall find a less crude inequality. Without loss of generality, assume that $\sigma(1) = 1$ and $\sigma(2) = 2$: that is, A_1^i is mapped into V_1 and A_2^i into V_2 . Write $A_1^i = \{y_1, \dots, y_{2s^{r-1}}\}$, and $A_2^i = \{z_1, \dots, z_{2s^{r-1}}\}$, where $y_j z_j$, $j \in [2s^{r-1}]$, are the edges of a perfect matching in H^i (which is guaranteed by Proposition 3.1). For ease of notation, let $f = f_i$. Then

$\mathbb{P}(E_i(A, B))$ is at most

$$\begin{aligned}
& \prod_{j=1}^{2s^{r-1}} (1 - \mathbb{P}(f(z_j) \in B \setminus \{f(z_1), \dots, f(z_{j-1})\} \text{ and } f(y_j) \in A \setminus \{f(y_1), \dots, f(y_{j-1})\})) \\
&= \prod_{j=1}^{2s^{r-1}} (1 - \mathbb{P}(f(z_j) \in B \setminus \{f(z_1), \dots, f(z_{j-1})\})) \mathbb{P}(f(y_j) \in A \setminus \{f(y_1), \dots, f(y_{j-1})\})) \\
&\leq \left(1 - \left(\frac{\delta n/2 - 2s^{r-1}}{n}\right)^2\right)^{2s^{r-1}} \leq \exp\left(-\frac{s^{r-1}\delta^2}{8}\right),
\end{aligned}$$

where the last inequality follows by recalling that $\frac{\delta}{4}n > 2s^{r-1}$. It follows that

$$\mathbb{P}(E(A, B)) \leq \exp\left(-\frac{s^{r-1}\delta^2}{8}\right)^{\frac{t}{\binom{r}{2}}-1} \leq \exp\left(-\frac{ts^{r-1}\delta^2}{8r^2}\right),$$

as there are at least $\frac{t}{\binom{r}{2}} - 1$ graphs in $\mathcal{F}(\{1, 2\})$. And finally,

$$\mathbb{E}X \leq 4^n \exp\left(-\frac{ts^{r-1}\delta^2}{8r^2}\right) = \exp\left(n \log 4 - \frac{ts^{r-1}\delta^2}{8r^2}\right).$$

Recalling our choices of s and t , we have $\mathbb{E}X < 1$. Therefore there exists a choice of f_1, \dots, f_t so that $X = 0$. We let $H = H(f_1, \dots, f_t)$ be a graph defined by such a choice and then define G by

$$E(G) = \{x_i y : y \in f_i(H^i), i \in [t]\} \cup E(T_r(n-t)) \setminus E(H) \cup E',$$

where E' is a maximal collection of edges added from $V_{r+1}^{(2)} \cup \{xy : x \in V_1 \cup \dots \cup V_r, y \in V_{r+1}\}$ so that G remains K_{r+1} -free. It is easy to check the resulting graph is $(r+1)$ -saturated. Adding any edges between V_{r+1} and $V_i, i \in [r]$, or inside V_{r+1} clearly forms a K_{r+1} by construction. The addition of any edge between the classes completes a K_{r+1} as it must be contained in one of the $V(f_i(H^i)), i \in [t]$ and hence completes a K_{r+1} by Proposition 3.1. The addition of an edge within one of the classes completes a K_{r+1} with $r-1$ of the vertices $\{v_1, \dots, v_r\}$.

We quickly see that there can be no complete bipartite graph of size $K_{\delta n, \delta n}$ in G as (for n sufficiently large) this would imply the existence of a $K_{\delta n/2, \delta n/2}$ as a subgraph of $G[V_1, \dots, V_r]$. But this is clearly forbidden by the choice of H . Hence, we clearly cannot have a balanced complete r -partite subgraph on more than $\delta r n$ vertices in G .

It remains to show that there are many edges in G . Using the estimate $e(G_{r,s}) \leq 4(r-1)s^r$ given by Proposition 3.1, we see that

$$\begin{aligned}
e(G) &> t_r(n-t) - t \cdot e(G_{r,s}) \geq t_r(n) - tn - t4(r-1)s^r \\
&\geq t_r(n) - \frac{128r^3}{\delta^2} n^{\frac{r+1}{r}},
\end{aligned}$$

and so the result holds with $C = C(r, \delta) = \frac{128r^3}{\delta^2}$. □

5 Final Remarks

Recall that $g_r(n, m)$ is defined to be the maximum number of vertices that one is required to remove from an n -vertex, $(r+1)$ -saturated graph with at least $t_r(n) - m$ edges, so that the remaining graph is complete r -partite. Combining Theorems 1.2 and 3.3 we see that (under some conditions on n and ε)

$$(\gamma_r + o(1))\varepsilon n^{2-\frac{1}{r}} \leq g_r(n, \varepsilon n^2) \leq C_r \varepsilon n^{2-\frac{1}{r}},$$

for some constants γ_r, C_r depending on r . We have not made any effort here to determine the best constant C_r that results from our proof. We conjecture that the lower bound is correct, in a suitable range of parameters. Let us write $f(n) \ll g(n)$ if $f(n)/g(n) \rightarrow 0$ as $n \rightarrow \infty$.

Conjecture 5.1. *Let $\varepsilon = \varepsilon(n)$ be a function satisfying $n^{-1} \ll \varepsilon(n) \ll n^{-\frac{r-1}{r}}$. Then*

$$g_r(n, \varepsilon n^2) = (\gamma_r + o(1))\varepsilon n^{2-\frac{1}{r}},$$

where $\gamma_r = \frac{1}{2} \left(\frac{4}{(r-1)r^{r-1}} \right)^{1/r}$ is the constant from the construction given in Theorem 3.3.

We actually conjecture that the construction given in Theorem 3.3 is indeed extremal for $g_r(n, m)$, for m in the range given above.

It is natural to consider the order of the largest *balanced* complete r -partite subgraph in $(r+1)$ -saturated graphs with at least $t_r(n) - m$ edges in the range $m \sim n^{\frac{r+1}{r}}$. We propose the following problem regarding the function $g_r^*(n, m)$, the balanced analogue of $g_r(n, m)$.

Problem 5.2. *Determine $g_r^* \left(n, Cn^{\frac{r+1}{r}} \right)$, for each $C \in \mathbb{R}^+$ and sufficiently large n .*

Theorem 1.3 shows that $g_r^*(n, Cn^{\frac{r+1}{r}}) \geq \left(1 - \frac{c'r^{5/2}}{C^{1/2}} \right) n$ for n sufficiently large, but we have no nontrivial upper bounds for $g_r^*(n, Cn^{\frac{r+1}{r}})$.

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